Problem Pages

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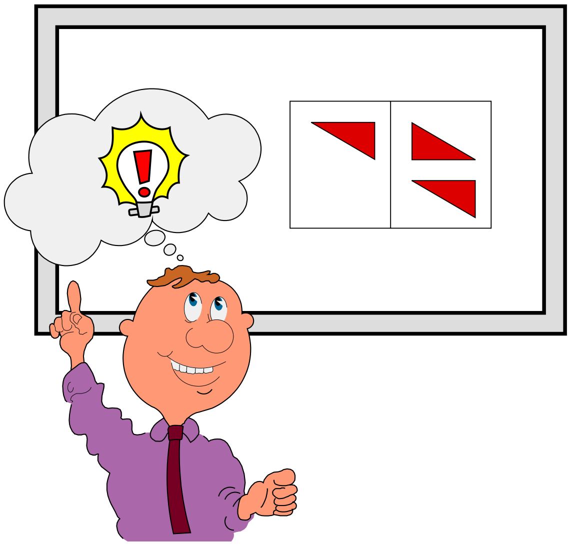


A photocopiable book of thought-provoking mathematics problems for sixth form and upper secondary school students.

Edited by
Charlie Stripp and
Steve Drape

The Mathematical Association

Introduction

This is a book of over sixty problems, together with suggested solutions. The problems are all accessible to students on A level or Scottish Higher courses, although many can be solved using mathematics covered lower down the school.

The problems are designed to stimulate interest and we hope that anyone in danger of being bored by Mathematics will be fascinated to find surprisingly simple and elegant solutions to what look like involved questions − solutions often so short and neat that they will stick in the memory. All of the problems encourage the development of problem-solving skills and should 'hook' students into hunting for a solution, helping them to develop the most important problem-solving skill of all − tenacity. Many of the problems require a degree of lateral thinking and should help to develop a sense of wonder at the power of Mathematics − when you know how! Questions which one cannot see how to do at first, or at all, yield solutions to surprise and delight. Very few solutions require more than a dozen lines and many need only five or six. We would not claim that many of the problems are new, but they make a fine collection for use in a variety of situations.

It is intended that the problems and solutions should be photocopied to allow their use to be as flexible as possible. Permission is given by The Mathematical Association to allow purchasers to make photocopies for use in their institutions.

Acknowledgements

This book of problems has been produced by the A and A/S level Subcommittee of the Teaching Committee of The Mathematical Association, and has been edited by Stephen Drape and Charlie Stripp.

The members of the Subcommittee who have contributed to the book are:

Barbara Cullingworth
Stephen Drape
Rosemary Emanuel
David Forster
Doug French
Christine Lawley
Jenny Orton
Charlie Stripp
Sally Tavemer
Peter Thomas
Marion Want

Other publications by this group include:

PIG and Other Tales, a book of mathematical readings with questions, suitable for sixth fomers.

Are You Sure? − Learning about Proof, a book of ideas for teachers of upper secondary school students.

Using this book

The book can provide a weekly problem over a two year cycle.

The problems can be used in a variety of ways. A method which has been tried successfully is to post up a problem each week in the classroom and invite students to work on it in their own time. Working together and discussing the problems should certainly be encouraged. The suggested solution can be posted up the following week, together with a new problem. Students' solutions can be displayed and alternative correct solutions can be a rich source of discussion. Many of the problems have a link to the standard curriculum, so they can be incorporated as an enhancement of normal teaching. The problems could form the basis of a student competition.

On the contents page each problem is given a level of difficulty ranging from 0, which is accessible to a GCSE level pupil, to 3 which is hard and will require a degree of inspiration. The problems are not presented in order of difficulty and are suitable for use in any order. Any difficulty rating is inevitably subjective and users may well disagree with some of our judgements.

Contents

Problem

Alphabetical Algebra

Suppose that

$$a = 1, b = 2, \ldots, z = 26$$

Evaluate the expression below:

$$(n - a)(n - b)\ldots(n - z)$$

Solution

Alphabetical Algebra

The expression simplifies to 0.

Any expression which has zero as a factor must equal zero:

$$(n - n)$$

(Did you multiply out?!!)

Problem

Suppose that

$$a = 1, b = 2, \ldots, z = 26$$

Evaluate the expression below:

$$(n - a)(n - b)\ldots(n - z)$$

Problem

How many primes?

Consider the nine-digit numbers formed by using each of the digits 1 to 9 once and only once.

e.g. 145673928

938267145

How many of these numbers are prime?

Solution

How many primes?

There are no prime numbers.

Add up the digits of each of the numbers and you get a total of 45 − which is divisible by 9.

Any number whose digit sum is divisible by 9 is itself divisible by 9 [can you prove this?].

This means that each of the 9! (362880) nine digit numbers is divisible by 9.

Therefore they are not prime.

Problem

Consider the nine-digit numbers formed by using each of the digits 1 to 9 once and only once.

e.g. 145673928

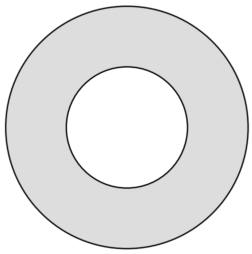
938267145

How many of these numbers are prime?

Problem

Just one dimension

Look at the shape below (called an *annulus*).



Find the single measurement from which the area of the annulus can be calculated.

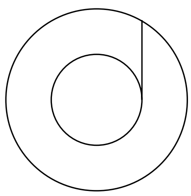
Solution

Just one dimension

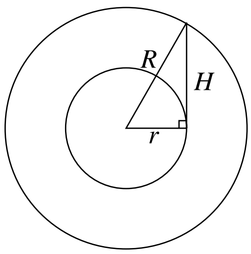
Let the small circle have radius r and the larger have radius R.

The shaded area is $\pi\left(R^2 - r^2\right)$.

How can this be obtained using one measurement?

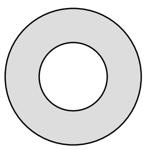


Draw the line shown, which is a tangent to the inner circle, and represent its length by H.



The triangle is right-angled since the tangent is perpendicular to a radius. So, by Pythagoras, $H^2 = R^2 - r^2$ and the area is πH^2.

Problem



Look at the shape alongside (called an annulus).

Find the single measurement from which the area of the annulus can be calculated.

Problem

Neighbours

Show that the sum of 3 consecutive numbers in an arithmetic sequence is three times the middle term.

*(An **arithmetic** sequence is one in which each pair of consecutive terms has a common difference.)*

Solution

Neighbours

The 3 numbers can be written as:

$$a \qquad a + d \qquad a + 2d$$

where a is the first term and d is the common difference.

The sum is equal to

$$a + (a + d) + (a + 2d)$$

$$= 3a + 3d$$

$$= 3(a + d)$$

Problem

Show that the sum of 3 consecutive numbers in an arithmetic sequence is three times the middle term.

*(An **arithmetic** sequence is one in which each pair of consecutive terms has a common difference.)*

Problem

Adding Fractions

If you add the numerators and denominators of two distinct fractions, then the resulting fraction lies between the two original fractions.

$$\text{e.g. } \frac{1}{2} \text{ and } \frac{1}{3} \text{ give } \frac{1+1}{2+3} = \frac{2}{5}$$

$$\text{and } \frac{2}{5} \text{ lies between } \frac{1}{2} \text{ and } \frac{1}{3}.$$

Prove that, if

$$\frac{a}{b} < \frac{c}{d}, \text{ then } \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

(a, b, c and d are positive integers).

Solution

Adding Fractions

Suppose that $\frac{a}{b} > \frac{a+c}{b+d}$, then $a(b+d) > b(a+c)$ and hence $ad > bc$.

This is a contradiction, since $\frac{a}{b} < \frac{c}{d}$ implies that $ad < bc$. Hence $\frac{a}{b} < \frac{a+c}{b+d}$.

$\frac{a+c}{b+d} < \frac{c}{d}$ can be proved similarly.

Problem

If you add the numerators and denominators of two distinct fractions, then the resulting fraction lies between the two original fractions.

Prove that, if

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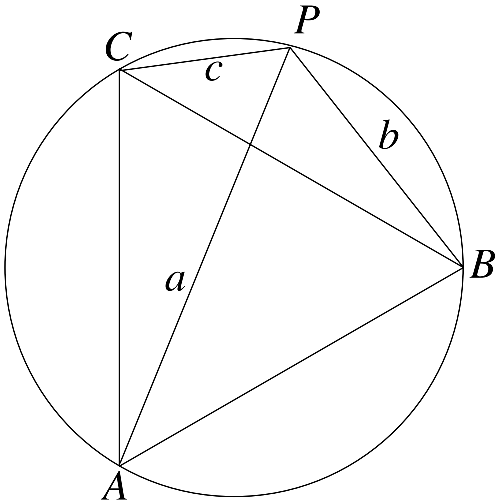
Problem

Van Schooten



ABC is an equilateral triangle inscribed in a circle.

P is a point on the minor arc BC.

If $AP = a$, $BP = b$, $CP = c$, prove $a = b + c$ (Van Schooten's Theorem).

Solution

Van Schooten

$\angle APB = \angle APC = 60°$ (angles in the same segment)

Similarly, $\angle APC = 60° = \angle APC$.

Applying the cosine rule to triangles *APB* and *APC*:

Since $\cos 60° = \frac{1}{2}$,

$$AB^2 = a^2 + b^2 - ab \text{ and } AC^2 = a^2 + c^2 - ac.$$

Since $AB = AC$, $b^2 - ab = c^2 - ac$

$$b^2 - c^2 = ab - ac$$

$$(b - c)(b + c) = a(b - c)$$

$$b + c = a, \quad \text{provided } b \neq c.$$

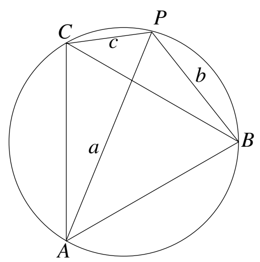
In the case where $b = c$, the two triangles, *APB* and *ABC* are right-angled and $b = c = \frac{1}{2}a$, so the result still holds.

Problem

ABC is an equilateral triangle inscribed in a circle.

P is a point on the minor arc *BC*.

If $AP = a$, $BP = b$, $CP = c$, prove $a = b + c$ (Van Schooten's Theorem).



Problem

3D Pythagoras

Consider using Pythagoras Theorem in 3-dimensions.

For this, we get the general result that

$$a^2 + b^2 + c^2 = d^2.$$

One set of numbers that will satisfy this, is 2, 3, 6, 7 since $2^2 + 3^2 + 6^2 = 7^2$.
Another set is 5, 6, 30, 31 since $5^2 + 6^2 + 30^2 = 31^2$.

Show algebraically that, if you choose two consecutive whole numbers and their product they will form the three smallest numbers of such a set of numbers and say what the largest will be.

Solution

3D Pythagoras

Let the three numbers be n, $n + 1$ and $n(n + 1)$.

Squaring and adding gives

$$\begin{aligned} n^2 + (n + 1)^2 + n^2(n + 1)^2 &= n^2 + n^2 + 2n + 1 + n^4 + 2n^3 + n^2 \\ &= n^4 + 2n^3 + 3n^2 + 2n + 1 \\ &= \left(n^2 + n + 1\right)^2 \end{aligned}$$

so these numbers fit the pattern.

As $n^2 + n + 1 = n(n + 1) + 1$, the largest number will always be one more than the product of the two smallest.

Problem

Consider using Pythagoras Theorem in 3-dimensions.

For this, we get the general result that

$$a^2 + b^2 + c^2 = d^2.$$

One set of numbers that will satisfy this, is 2, 3, 6, 7 since $2^2 + 3^2 + 6^2 = 7^2$.

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Show algebraically that, if you choose two consecutive whole numbers and their product they will form the three smallest numbers of such a set of numbers and say what the largest will be.

Problem

Build a wall

To build a certain wall, a supervisor knows that these builders work at the following rates:

a) Ali & Bill take 12 days
b) Ali & Charlie take 15 days
c) Bill & Charlie take 20 days

Assuming that the rates at which builders work are not affected by their companion:

1. How long would it take each of them working alone to build the wall?
2. How long would it take to build the wall if they all worked together?

Solution

Build a wall

Let a be the number of days it takes Ali to build the wall, so Ali builds $1/a$ of the wall each day.

Similarly, Bill builds $1/b$ and Charlie builds $1/c$ each day.

a) Since Ali and Bill build 1/12 of the wall each day $1/a + 1/b = 1/12$.

Similarly, $1/a + 1/c = 1/15$

and $1/b + 1/c = 1/20$.

Solving these three equations gives: $a = 20, b = 30, c = 60$.

b) Suppose that, working together, Ali, Bill & Charlie take x days,

then $1/x = 1/20 + 1/30 + 1/60$

$1/x = 3/60 + 2/60 + 1/60 = 6/60$

$x = 10$

So if they all work together they will take 10 days to build the wall.

Problem

To build a certain wall, a supervisor knows that these builders work at the following rates:

a) Ali & Bill take 12 days

b) Ali & Charlie take 15 days

c) Bill & Charlie take 20 days

Assuming that the rates at which builders work are not affected by their companion:

1. How long would it take each of them working alone to build the wall?
2. How long would it take to build the wall if they all worked together?

Problem

Sums of Sequences

Consider the formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

and

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Explain why the results of these are always integers.

Solution

Sums of Sequences

The first result is trivial. If n is odd then $n + 1$ is even and divides exactly by 2 and, if n is even, it divides exactly by 2 so, in either case the product divides by 2 and there is no remainder.

The second result will always divide by 2 for the same reason so it needs to be shown that it also divides by 3.

If n is a multiple of 3 or if $n + 1$ is a multiple of 3 then the whole product divides by 3.

If neither n nor $n + 1$ divide by three then $n - 1$ must.

But $2n + 1 = 2(n - 1) + 3$, both parts of which are divisible by three so the product is again divisible by 3.

Problem

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Explain why the results of these are always integers.

Problem

Happy Birthday!

What is the smallest number of people that it is necessary to have in a group so that the probability of at least two of them sharing the same birthday is greater than ½?

Solution

Happy Birthday!

The answer is 23.

Let n be the number of people in the group.

P(at least one pair with the same birthday)

$$= 1 - \text{P(no pairs)}$$

For one pair, P(different birthdays) = $\frac{364}{365}$

P(no pairs) = $\left(\frac{364}{365}\right)^p$ where $p = {}^nC_2 = \frac{n!}{(n-2)n!}$

(i.e. number of ways of choosing a pair)

We want P(no pairs) $\leqslant \frac{1}{2}$ (so P(at least one) $> \frac{1}{2}$).

$$\left(\frac{364}{365}\right)^p \leqslant \frac{1}{2}$$

Taking logs $\quad p \log \frac{364}{365} \leqslant \log \frac{1}{2}$

$$p \geqslant 252 \text{ as } \log \frac{364}{365} \leqslant 0.$$

$$ {}^{23}C_2 = 253, \text{ so } n = 23.$$

Problem

What is the smallest number of people that it is necessary to have in a group so that the probability of at least two of them sharing the same birthday is greater than ½?

Problem

Primes

A *prime* number is a number which only has 2 factors.

Show that all prime numbers (except 2 and 3) can be written in the form

$$6n \pm 1.$$

Solution

Primes

Let p be a number ($p > 3$) and n be an appropriate integer.

Case 1: If $p = 6n$, then p would have a factor of 6 and so p could not be prime.

Case 2: If $p = 6n + 2$, then p would have a factor of 2 and so p could not be prime.

Case 3: If $p = 6n + 3$ then p would have a factor of 3 and so p could not be prime.

Case 4: If $p = 6n - 2$ then, as with Case 2, p would have a factor of 2 and so it could not be prime.

Hence, p could be prime only if it was of the form

$$6n + 1 \text{ or } 6n - 1.$$

(Note: the converse is *not* true, $25 = 6 \times 4 + 1$ is not prime.)

This is an example of *proof by exhaustion* − all the possible cases have been considered.

Problem

A *prime* number is a number which only has 2 factors.

Show that all prime numbers (except 2 and 3) can be written in the form $6n \pm 1$.