MATH(AH)10

NATIONAL QUALIFICATIONS 2010

TIME: 3 HOURS

MATHEMATICS ADVANCED HIGHER

Covering units 1 & 2

Read Carefully

1. Calculators may be used in this paper.

2. Candidates should answer **all** questions.

3. **Full credit will be given only where the solution contains appropriate working**.

Answer all the questions

Marks

1. *(a)* $y = 3x^4 \sin^{-1} x$, where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Find an expression for $\frac{dy}{dx}$. **3**

(b) Find an expression for $f'(x)$ when $f(x) = \frac{3 - x^5}{e^{2x}}$. **3**

2. Use Gaussian elimination to solve this system of equations

$$\begin{aligned} x + y - 3z &= -4 \\ 3x - 3y + 4z &= 21 \\ x + 3y + 4z &= 2 \end{aligned}$$

5

3. A curve is defined by the parametric equations

$$x = 3t^2 + 8 \quad \text{and} \quad y = 7 - 3t - t^2$$

for all t. Find the equation of the tangent to the curve when $t = 2$. **5**

4. An arithmetic series and a geometric series both have terms $u_3 = 6$ and $u_5 = 14$. A second geometric series with the term $v_2 = 135$ and common ratio r has a sum to infinity equal to the sum of the first 20 terms of the arithmetic series.
Calculate the possible values for the common ratio, r, of the second geometric series. **6**

5. A curve is defined by the equation $y = \frac{x^2 + 4}{(x-2)^2}$, $x \neq 2$.

(a) Express this equation in form $y = A + \frac{Bx}{(x-2)^2}$. **1**

(b) (i) Write down the equations of the asymptotes to the curve. **2**

(ii) Obtain the stationary point(s) of the curve and justify their natures. **5**

(c) Sketch the curve, showing all the features found in part (*b*). **2**

6. Prove by induction that $8^n - 1$ is divisible by 7 for all positive integers n. **5**

Marks

7. Use the substitution $u = 3 - x^2$ to evaluate $\int_0^1 \frac{x}{\sqrt[3]{3 - x^2}}\,dx$. **5**

8. *(a)* Show that $z = -2 - 2i$ is a root of the equation

$$z^3 + z^2 - 4z - 24 = 0$$ **2**

(b) Obtain the other roots of the equation. **2**

9. Express $\frac{8x^2 + x + 5}{x(x^2 + 1)}$ in the form $\frac{A}{x} + \frac{Bx + C}{x^2 + 1}$, stating the values of the constants *A*, *B* and *C*. **3**

Hence determine an expression for $\int \frac{8x^2 + x + 5}{x(x^2 + 1)}\,dx$. **4**

10. Given that $\frac{dy}{dx} = 3y\sec^2 x$ and $y = 240$ when $x = \frac{\pi}{4}$, find an expression for *y* in terms of *x* only. **5**

11. Prove by contradiction that if *a* is odd then $(a + 3)^2$ must be even, where *a* is a positive integer. **3**

[Turn over

Marks

12. Let $z = \sqrt{2}\cos\theta + i\sqrt{2}\sin\theta$.

(a) Use de Moivre's theorem to find an expression for z^3. **1**

(b) Use the binomial expansion to find another expression for z^3. **3**

(c) Using the results from parts (*a*) and (*b*) show that

$$\frac{\cos 3\theta}{\cos^3\theta} = a + b\tan^2\theta$$

stating the values of the constants *a* and *b*. **3**

13. Use integration by parts to obtain the value of $\int_0^3 x^2 e^{4x}\,dx$. **6**

14. A function *f* is defined by the equation $y = (1+x)^3(x+2)^{-3}e^{2x}$.

Use logarithmic differentiation to obtain an expression for $\frac{dy}{dx}$ in terms of *x*. **3**

Hence find the equation of the tangent to the curve when $x = 0$. **2**

15. Calculate $\sum_{r=7}^{34}(3k+7)$. **4**

16. A function is defined on a suitable domain as $xy + y^2 = -4$.

(a) Find an expression for $\frac{dy}{dx}$. **3**

(b) Hence find an equation of a tangent to the curve at $x = -4$. **3**

(c) Determine an expression for $\frac{d^2y}{dx^2}$ in terms of *x* and *y* only. **4**

17. Use the substitution $u^2 = (3x^2 - 1)^2$ to obtain $\int_0^{\frac{1}{\sqrt{3}}} \frac{6x}{\sqrt{6x^2 - 9x^4}}\,dx$. **7**

[*END OF QUESTION PAPER*]

TOTAL **100**

Additional Questions for Unit 3 *Marks*

A. The points A(1, 3, 0), B(–2, 0, 5) and C(2, –3, –1) all lie in the plane $\prod$.

(a) Calculate the equation of plane $\prod$. **4**

(b) Calculate the point of intersection between the line $L: \frac{x+3}{2} = y-5 = \frac{-z}{3}$ and the plane $\prod$ and the size of the angle between L and $\prod$. **5**

B. Find the Maclaurin expansion for $f(x) = e^{\sin x}$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ as far as the x^4 term. **5**

C. Given that for matrix A, $A^2 = 5A - 2I$ where I is the corresponding identity matrix, find the integers x and y such that

$$A^4 = xA + yI$$ **4**

D. Obtain the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 8\cos x$$ **7**

Hence find the particular solution given that $\frac{dy}{dx} = 0$ and $y = 2$, when $x = 0$. **3**

[*END OF ADDITIONAL QUESTIONS*]

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[MATH(AH)MS - 2010]

NATIONAL
QUALIFICATIONS
2010

Marking Instructions

ADVANCED
HIGHER
MATHEMATICS

The security of this examination paper requires that it is withdrawn from candidates after the examination and also after any discussion of the candidates' results. This will ensure that the paper continues to be secure for your centre and others during presentation year 2009/2010. Any appeals made based on this paper will assume that these security precautions are in place.

Advanced Higher Mathematics

Marking Instructions

Distribution of marks
Candidates will be expected to answer all of the questions. There will be a total of 100 marks for the paper.

The below suggested marking thresholds are based on an unaltered paper for units 1 and 2.

If inserting unit 3 questions then the below marking thresholds may only be used if:

1] the total number of A marks and the total number of B marks is the same or greater
2] each of the three units has at least 30% of the marks

If either or both of the above criteria are not met, the cutoffs should be adjusted *upwards*

Suggested Marking Thresholds

Mark	**Grade**
90%	A1
75%	A
63%	B
50%	C
45%	D

No	Analysis: Unit / Outcome	Analysis: Marks at levels A/B	Analysis: Marks at levels C	Question	Illustrations of evidence for awarding each mark	Marks
1. *(a)*	1b Differentiation		3	$y = 3x^4 \sin^{-1} x$, where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Find an expression for $\frac{dy}{dx}$.	• Use the product rule $\frac{dy}{dx} = \frac{d}{dx}\left(3x^4\right)\left(\sin^{-1} x\right) + \left(3x^4\right)\frac{d}{dx}\left(\sin^{-1} x\right)$ **1** • Know the derivative of $\sin^{-1} x$ $12x^3 \cdot \sin^{-1} x + 3x^4 \cdot \boxed{\frac{1}{\sqrt{1-x^2}}}$ $12x^3 \cdot \sin^{-1} x + \frac{3x^4}{\sqrt{1-x^2}}$ **1** • Accuracy **1**	**3**
(b)	1b Differentiation	1	2	Find an expression for $f'(x)$ when $f(x) = \frac{3-x^5}{e^{2x}}$.	• Use the quotient rule $f'(x) = \frac{\frac{d}{dx}\left(3-x^5\right)\left(e^{2x}\right) - \left(3-x^5\right)\frac{d}{dx}\left(e^{2x}\right)}{\left(e^{2x}\right)^2}$ **1** • Use the chain rule to differentiate e^{2x} $\frac{-5x^4 \cdot e^{2x} - \left(3-x^5\right) \cdot \boxed{\left(2e^{2x}\right)}}{e^{4x}}$ $\frac{2x^5 - 5x^4 - 6}{e^{2x}}$ **1** • Accuracy **1**	**3**

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
2.	1e Matrices		5	Use Gaussian elimination to solve this system of equations $x + y - 3z = -4$ $3x - 3y + 4z = 21$ $x + 3y + 4z = 2$	• Express system of equations as a matrix in augmented form $\begin{pmatrix} 1 & 1 & -3 & -4 \\ 3 & -3 & 4 & 21 \\ 1 & 3 & 4 & 2 \end{pmatrix}$ **1** • Begin elementary row operations so that $a_{21} = 0$ and $a_{31} = 0$ $\begin{matrix} \\ R2 \to R2 - 3R1 \\ R3 \to 3R3 - R2 \end{matrix} \begin{pmatrix} 1 & 1 & -3 & -4 \\ 0 & -6 & 13 & 33 \\ 0 & 12 & 8 & -15 \end{pmatrix}$ **1** • Repeat elementary row operations until matrix is in upper triangular form $R3 \to R3 + 2R2 \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & -6 & 13 & 33 \\ 0 & 0 & 34 & 51 \end{pmatrix}$ **1** • Solve for z using row 3. $34z = 51 \quad \therefore z = \frac{3}{2}$ **1** • Back substitute to find values for y then x. $\therefore \; y = -\frac{9}{4} \; and \; x = \frac{11}{4}$ **1**	**5**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
3.	2a Differentiation	4	2	A curve is defined by the parametric equations $$x = 3t^2 + 8 \text{ and } y = 7 - 3t - t^2$$ for all t. Find the equation of the tangent to the curve when $t = 2$.	• Calculate $\frac{dx}{dt}$ *and* $\frac{dy}{dt}$ $\frac{dx}{dt} 6t$ *and* $\frac{dy}{dt} = -3 - 2t$ **1** • Know how to find $\frac{dy}{dx}$ $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ **1** • Accuracy of $\frac{dy}{dx}$ $\frac{dy}{dx} = \frac{-3-2t}{6t}$ **1** • Find a point on the line When $t = 2$, $(x, y) = (20, -3)$ **1** • Find the gradient of the line $m = \frac{dy}{dx} = -\frac{7}{12}$ **1** • Use the point-gradient formula to state the equation of the tangent $y + 3 = -\frac{7}{12}(x - 20)$ **1**	**6**

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
4.	2d Sequences & Series	3	3	An arithmetic series and a geometric series both have terms $u_3 = 6$ and $u_5 = 14$. A geometric series with the term $v_2 = 135$ and common ratio r has a sum to infinity equal to the sum of the first 20 terms of the arithmetic series. Calculate the possible values for the common ratio, r, of the geometric series.	• Find the common difference and initial term of the arithmetic series. $u_6 = u_3 + 2d = 6 + 2d = 14$ $\therefore d = 4 \; and \; u_1 = a = -2$	1
					• Find the sum of the first 20 terms of the arithmetic series $S_{20} = \frac{n}{2}\left(2a + (n-1)d\right)$ $= \frac{20}{2}\left(-4 + 19 \times 4\right)$ $= 720$	1
					• Express v_1 in terms of v_2 and r $v_2 = ar = 135 \;\; \therefore a = \frac{135}{r}$	1
					• Substitute into the formula for the sum to infinity terms of a geometric series $S_\infty = \frac{a}{1-r}$ $= \frac{135}{r(1-r)}$ $= 720$	1
					• Create equation for r	1

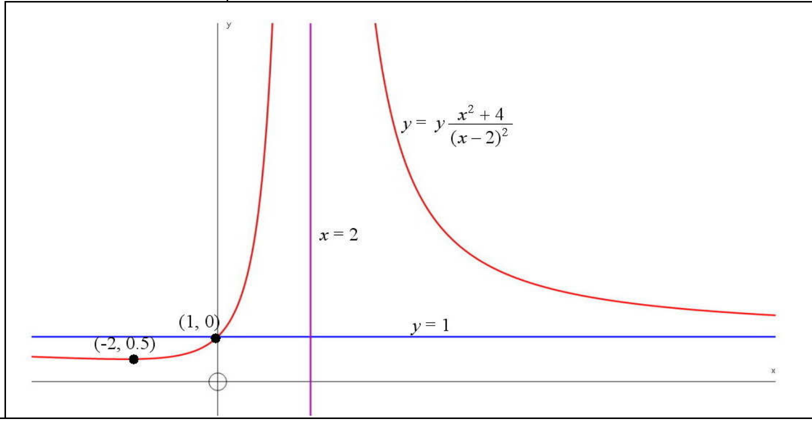
No	Analysis			Question	Illustrations of evidence for awarding each mark		Marks
	Unit / Outcome	Marks at levels					
4.*(Cont)*					• Solve for r $r^2 - r + \frac{3}{16} = 0$ $r = \frac{1 \pm \frac{1}{2}}{2}$ $= \frac{1}{4}, \frac{3}{4}$	1	6
5.				A curve is defined by the equation $y = \frac{x^2+4}{(x-2)^2}$, $x \neq 2$.			
(a)	1a Algebra	1		Express this equation in form $y = A + \frac{B}{(x-2)^2}$.	• Division $x^2 - 4x + 4 \overline{)\, x^2 + 0x + 4}$ quotient 1 $x^2 - 4x + 4$ $4x$ $\therefore\ y = 1 + \frac{4x}{(x-2)^2}$		1
(b) (i)	1d Functions		2	Write down the equations of the asymptotes to the curve.	• State the vertical asymptote $x = 2$ • State the non-vertical asymptote $y = 1$	1 1	2

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
5. (Cont) *(b)* (ii)	1b Differentiation		5	Obtain the stationary point(s) of the curve and justify their natures.	• Know to use quotient rule to calculate $\frac{dy}{dx}$ $\frac{dy}{dx} = \frac{4(x-2)^2 - 4x \cdot 2(x-2)^1}{(x-2)^4}$ **1** • Calculate $\frac{dy}{dx}$ accurately $\frac{dy}{dx} = -\frac{4(x+2)}{(x-2)^3}$ **1** • Create equation for $\frac{dy}{dx} = 0$ • Find stationary points Stationary points occur when $x + 2 = 0$ $x = -2, y = \frac{1}{2}$ **1** **1** • Determine their nature x: →, −2, → $\frac{dy}{dx}$: −ve, 0, +ve Shape: \, __, / $(-2, \frac{1}{2})$ is a minimum turning point. **1**	5

Nature table:

x	→	−2	→
$\frac{dy}{dx}$	−ve	0	+ve
Shape	\	__	/

No	Analysis: Unit / Outcome	Marks at levels: A/B	C	Question	Illustrations of evidence for awarding each mark	Marks
5. (Cont) *(c)*	1d Functions		2	Sketch the curve, showing all the features found in part (*b*).	• Graph: asymptotes and general shape **1** • Graph: stationary point **1** 	**2**

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark		Marks
	Unit / Outcome	Marks at levels					
		A/B	C				
6.	2e Proof		5	Prove by induction that $8^n - 1$ is divisible by 7 for all positive integers *n*.	• Show true for $n = 1$ and assume true for $n = k$	**1**	
					• For $n = 1$, $8^1 - 1 = 7$ which is divisible by 7. Assume true for $n = k$ $\therefore 8^k - 1 = 7a$ $\quad a > 0$ $\therefore 8^k = 7a + 1$	**1**	
					• Consider the case where $n = k + 1$ For $n = k + 1$, $8^{k+1} - 1 = \ldots\ldots$	**1**	
					• Manipulate $= 8^k \cdot 8 - 1$ $= (7a + 1) \cdot 8 - 1$ $= 56a + 7$ $= 7(8a + 1)$		
					• Prove which is divisible by 7	**1**	
					• Conclusion Hence, if true for $n = k$ then also true for $n = k + 1$ and since true for $n = 1$, true $\forall n$.	**1**	**5**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
7.	1c Integration		5	Use the substitution $u = 3 - x^2$ to evaluate $\int_0^1 \frac{x}{\sqrt[3]{3-x^2}}\,dx$.	• Find $\frac{du}{dx}$ $\frac{du}{dx} = -2x \quad \therefore dx = -\frac{1}{2x}\,du$ when $x = 1$, $u = 2$ and when $x = 0$, $u = 3$. **1** • Substitute $\int_3^2 \frac{\cancel{x}}{\sqrt[3]{u}} \cdot \frac{-1}{2\cancel{x}}\,du$ **1** • Simplify $-\frac{1}{2}\int_3^2 u^{-\frac{1}{3}}\,du$ **1** • Integrate $-\frac{3}{4}\left[\sqrt[3]{u^2}\right]_3^2$ **1** • Evaluate $\frac{3}{4}\left(\sqrt[3]{9} - \sqrt[3]{4}\right)$ **1**	5

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
8. *(a)*	2c Complex numbers		2	Show that $z = -2 - 2i$ is a root of the equation $z^3 + z^2 - 4z - 24 = 0$	• Calculate z^2 and z^3 $z^2 = 8i$ *and* $z^3 = 16 - 16i$ Show that $f(-2-2i) = 0$ • $(16-16i) + (8i) - 4(-2-2i) - 24 = 0$	**2**
(b)	2c Complex numbers		2	Obtain the other roots of the equation.	• Know that if z is a root then so is $\bar{z}$ and $z\bar{z}$ If $-2-2i$ is a root then $\left(z-(-2-2i)\right) = \left((z+2)+(2i)\right)$ is a factor as is $\left((z+2)-(2i)\right)$ and $\left((z+2)+(2i)\right)\left((z+2)-(2i)\right) = z^2 + 4z + 8$ • Division and solution $z - 3$ $z^2 + 4z + 8 \overline{)z^3 + z^2 - 4z - 24}$ $z^3 + 4z^2 + 8z$ $-3z^2 - 12z - 24$ $-3z^2 - 12z - 24$ 0 $\therefore\ z = \underline{\underline{-2-2i}},\ \underline{\underline{2+2i}},\ \underline{\underline{3}}$	**2**

No	Analysis			Question	Illustrations of evidence for awarding each mark		Marks
	Unit / Outcome	Marks at levels					
		A/B	C				
9.	1a Algebra		3	Express $\frac{8x^2+x+5}{x(x^2+1)}$ in the form $\frac{A}{x}+\frac{Bx+C}{x^2+1}$, stating the values of the constants *A*, *B* and *C*.	• Equate and multiply through by $x(x^2+1)$ $\frac{8x^2+x+5}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$ Multiply through by $x(x^2+1)$ $8x^2+x+5=A(x^2+1)+Bx$	**1**	
					• Find the value of A Let $x=0,\quad 5=A+0\quad \therefore \underline{\underline{A=5}}$	**1**	
					• Substitute in the value of A and find B and C. $8x^2+x+5=5(x^2+1)+(Bx+C)x$ $3x^2+x=Bx^2+Cx$ $\therefore \underline{\underline{B=3}},\ \underline{\underline{C=1}}$	**1**	**3**
	1c & 2b Integration		4	Hence determine an expression for $\int \frac{8x^2+x+5}{x(x^2+1)}\,dx$.	• Create terms that can be integrated $\int\left(\frac{5}{x}+\frac{3x+1}{x^2+1}\right)dx=\int\left(\frac{5}{x}+\frac{3}{2}\cdot\frac{2x}{x^2+1}+\frac{1}{x^2+1}\right)dx$	**1**	
					• Integrate $\frac{1}{x}$ $5\ln x+....$	**1**	
					• Integrate $\frac{f'(x)}{f(x)}$ $....+\frac{3}{2}\ln\left\|x^2+1\right\|+....$	**1**	
					• Integrate $\frac{1}{1+x^2}$ $....+\tan^{-1}x+c$	**1**	**4**

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark		Marks
	Unit / Outcome	Marks at levels					
		A/B	C				
10.	2b Integration		5	Given that $\frac{dy}{dx} = 3y\sec^2 x$ and $y = 240$ when $x = \frac{\pi}{4}$, find an expression for y in terms of x only.	• Separate variables $\frac{1}{y}dy = 3\sec^2 x\ dx$ • Integrate $\int \frac{1}{y}\ dy = \int 3\sec^2 x\ dx$ $\ln y = 3\tan x + c$ • Exponential of each side $e^{\ln y} = e^{3\tan x + c}$ $let\ e^c = A$ $y = Ae^{3\tan x}$ • Substitute in values $240 = Ae^{3\tan\frac{\pi}{4}} = Ae^3$ $A = \frac{240}{e^3}$ • Expression for y $y = \frac{240}{e^3}e^{3\tan x}$	1 1 1 1 1	5
11.	2e Proof		3	Prove by contradiction that a is odd the $(a+3)^2$ must be even, where a is a positive integer.	• Assume that a is even $\therefore\ a = 2k$ • Calculate $(a+3)^2$ $(a+3)^2 = (2k+3)^2$ $= 4k^2 + 12k + 9$ $= 2(2k^2 + 6k + 4) + 1$ which is odd. • Conclusion Hence, the assumption must be false. Therefore, a is not odd. Therefore, a is even.	1 1 1	3

No	Analysis: Unit / Outcome	Analysis: Marks at levels A/B	Analysis: Marks at levels C	Question	Illustrations of evidence for awarding each mark	Marks
12.				Let $z = \sqrt{2}\cos\theta + i\sqrt{2}\sin\theta$.		
(a)	2c Complex numbers	1		Use de Moivre's theorem to find an expression for z^3.	• Expression for z^3 $z^3 = \sqrt{2}^3(\cos 3\theta + i\sin 3\theta)$ **1**	**1**
(b)	2c Complex numbers		3	Use the binomial expansion to find another expression for z^3.	$z^3 = \sqrt{2}^3(\cos\theta + i\sin\theta)^3$ • Numerical coefficients $= \sqrt{2}^3\left(\binom{3}{3}(\cos\theta)^3(i\sin\theta)^0 + \binom{3}{2}(\cos\theta)^2(i\sin\theta)^1 + \binom{3}{1}(\cos\theta)^1(i\sin\theta)^2 + \binom{3}{0}(\cos\theta)^0(i\sin\theta)^3\right)$ **1** • Correct terms $= \sqrt{2}^3(\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta)$ **1** • Simplify $= \sqrt{2}^3\left((\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta - \sin^3\theta)\right)$ **1**	**3**
(c)	2c Complex numbers	3		Using the results from parts (*a*) and (*b*) show that $\frac{\cos 3\theta}{\cos^2\theta} = a\cos\theta + b\tan^2\theta$ stating the values of the constants *a* and *b*.	• Equate Real parts $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ **1** • Divide $\frac{\cos 3\theta}{\cos^3\theta} = 1 - 3\frac{\sin^2\theta}{\cos^2\theta}$ **1** • Simplify and state solution $\frac{\cos 3\theta}{\cos^3\theta} = 1 - 3\tan^2\theta \quad \therefore a = 1,\ b = -3$ **1**	**3**

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
13.	2b Integration	6		Use integration by parts to obtain the value of $\int_0^3 x^2 e^{4x} dx$.	• First application done correctly $\int_0^3 e^{4x} \cdot x\ dx = \left[\frac{1}{4}e^{4x} \cdot x^2 - \int \frac{1}{4}e^{4x} \cdot 2x\ dx\right]_0^3$ **1** • Second application: terms 1 and 2 correct $= \left[\frac{1}{4}e^{4x} \cdot x^2 - \frac{1}{2}\left(\boxed{\frac{1}{4}e^{4x} \cdot x} - \int \frac{1}{4}e^{4x} \cdot 1dx\right)\right]_0^3$ **1** • Second application: terms 3 and 4 correct $= \left[\frac{1}{4}e^{4x} \cdot x^2 - \frac{1}{2}\left(\frac{1}{4}e^{4x} \cdot x - \boxed{\int \frac{1}{4}e^{4x} \cdot 1dx}\right)\right]_0^3$ **1** • Integrate $= \left[\frac{1}{4}e^{4x} \cdot x^2 - \frac{1}{8}e^{4x} \cdot x + \frac{1}{32}e^{4x}\right]_0^3$ **1** • Simplify $= \frac{1}{32}\left[e^{4x}\left(8x^2 - 4x + 1\right)\right]_0^3$ **1** • Evaluate $= \frac{1}{32}\left(61e^{12} - 1\right)$ $\approx 310251 \cdot 3$ **1**	**6**

No	Analysis			Question	Illustrations of evidence for awarding each mark		Marks
	Unit / Outcome	Marks at levels					
		A/B	C				
14.	2a Differntiation	5		A function f is defined by the equation $y = (1+x)^3(x+2)^{-3}e^{2x}$. Use logarithmic differentiation to obtain an expression for $\frac{dy}{dx}$ in terms of x.	• Use logarithms to simplify $\ln y = 3\ln(1+x) - 3\ln(x+2) + 2x$	**1**	
					• Differentiate $\frac{1}{y}\cdot\frac{dy}{dx} = \frac{3}{1+x} - \frac{3}{x+2} + 2$	**1**	
					• Expression for $\frac{dy}{dx}$: $\frac{dy}{dx} = \left(\frac{3}{1+x} - \frac{3}{x+2} + 2\right)\left((1+x)^3(x+2)^{-3}e^{2x}\right)$	**1**	**3**
				Hence find the equation of the tangent to the curve when $x = 0$.	• Find a point on the line and calculate the gradient. When $x = 0,\ y = \frac{1}{8}$; $m = \frac{dy}{dx} = \left(\frac{3}{1} - \frac{3}{2} + 2\right)\left(\frac{1}{8}\right) = \frac{7}{16}$	**1**	
					• Use the point-gradient formula correctly $y - \frac{1}{8} = \frac{7}{16}x$	**1**	**2**

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
15.	2d Sequences & Series		4	Calculate $\sum_{r=7}^{34}(3k+7)$.	• Identify *a* and *d* $a=10,\ \ d=3$ **1** • Find the sum of the first 6 terms $S_6=\sum_{r=1}^{6}(3k+7)$ $=\frac{n}{2}(2a+(n-10)d)$ $=\frac{6}{2}(20+5\times 3)$ $=105$ **1** • Find the sum of the first 34 terms $S_{34}=\sum_{r=1}^{34}(3k+7)$ $=\frac{34}{2}(20+33\times 3)$ $=2023$ **1** • Solution $\sum_{r=7}^{34}(3k+7)=2023-105$ $=1918$ **1**	**4**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
16				A function is defined on a suitable domain as $xy + y^2 = -4$.		
(a)	2a Differntiation		3	Find an expression for $\frac{dy}{dx}$.	• Implicit differentiation: use the product rule to differentiate *xy* $1 \cdot y + x \cdot \frac{dy}{dx} + ...$ • Use the Chain Rule to differentiate y^2 $.... + 2y \cdot \frac{dy}{dx} = 0$ • $y + \frac{dy}{dx}(x + 2y) = 0$ $\frac{dy}{dx} = -\frac{y}{x + 2y}$	**3**
(b)	2a Differntiation		3	Hence find an equation of a tangent to the curve at $x = -4$.	• Create an equation for *y* and solve $4y + y^2 = -4$ • When $x = -4$, $(y - 2)^2 = 0$ $y = 2$ • When $x = -4$, $y = 2$ $m = \frac{dy}{dx} = \frac{2}{-4 + 4} = \frac{2}{0}$ which is undefined so tangent must be vertical and equation is $x = -4$.	**3**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
16(cont)						
(c)	2a Differntiation	3		Determine an expression for $\frac{d^2y}{dx^2}$ in terms of *x* and *y* only.	• Use the quotient rule to get $\frac{d^2y}{dx^2}=\frac{-\frac{dy}{dx}(x+2y)+y(1+2(-\frac{dy}{dx}))}{(x+2y)^2}$ • Substitute in expression for $\frac{dy}{dx}$ $\frac{d^2y}{dx^2}=\frac{\left(\frac{y}{x+2y}\right)(x+2y)+y\left(1-2\frac{y}{x+2y}\right)}{(x+2y)^2}$ • Simplify $\frac{d^2y}{dx^2}=\frac{y+y-\frac{2y^2}{x+2y}}{(x+2y)^2}$ • Accuracy $\frac{d^2y}{dx^2}=\frac{2xy+2y^2}{(x+2y)^3}$	**4**

No	Analysis			Question	Illustrations of evidence for awarding each mark		Marks
	Unit / Outcome	Marks at levels					
		A/B	C				
17.	2b Integration	4	3	Use the substitution $u^2 = (3x^2 - 1)^2$ to obtain $\int_0^{\frac{1}{\sqrt{3}}} \frac{6x}{\sqrt{6x^2 - 9x^4}}\,dx$.	• Find $\frac{du}{dx}$ and convert limits $\frac{du}{dx} = 6x,\ u = 3x^2 - 1,\ 3x^2 = u + 1,\ dx = \frac{du}{6x}$ *when* $x = \frac{1}{\sqrt{3}},\ u = 0$ *and when* $x = 0,\ u = -1$	1	
					• Substitute $\int_{-1}^{0} \frac{6x}{\sqrt{2(u+1) - (u+1)^2}} \cdot \frac{du}{6x}$	1	
					• Simplify $\int_{-1}^{0} \frac{1}{\sqrt{2u + 2 - (u^2 + 2u + 1)}}\,du$	1	
					• Further simplification $\int_{-1}^{0} \frac{1}{\sqrt{1 - u^2}}\,du$	1	
					• Integrate $\left[\sin^{-1} u\right]_{-1}^{0}$	1	
					• Evaluate $\sin^{-1}(0) - \sin^{-1}(-1)$	1	
					• Solution $-\frac{3\pi}{2}$	1	7

Total 100 marks

[END OF MARKING SCHEME]

Additional Questions for unit 3

No	Analysis: Unit / Outcome	Marks at levels: A/B	Marks at levels: C	Question	Illustrations of evidence for awarding each mark	Marks
A	3a Vectors			The points A(1, 3, 0), B(–2, 0, 5) and C(2, –3, –1) all lie in the plane $\prod$.		
(a)			4	Calculate the equation of plane $\prod$.	• Vector product of two vectors in the plane to get a normal vector. $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -3 & -3 & 5 \\ 1 & -6 & -1 \end{vmatrix}$ **1** • Accuracy $\underset{\sim}{n} = \begin{pmatrix} 33 \\ 2 \\ 21 \end{pmatrix}$ **1** • Use scalar product $n \cdot \overrightarrow{AP} = \begin{pmatrix} 33 \\ 2 \\ 21 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-3 \\ z-0 \end{pmatrix}$ **1** • Solution $33x + 2y + 21z = 36$ **1**	**4**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
A.(Cont) *(b)*			5	Calculate the point of intersection between the line $L: \frac{x+3}{2} = y-5 = \frac{-z}{3}$ and the plane $\prod$ and the size of the angle between L and $\prod$.	• Converting equation of line into parametric form and substituting values in equation of plane $x = 2t-3,\ \ y = t+5,\ \ z = -3t \quad \therefore 5t = 125$ **1** • Point of intersection t = 25 so point of intersection is (47, 30, –75) **1** • Use scalar product correctly $\underset{\sim}{n} \cdot \underset{\sim}{l} = \begin{pmatrix} 33 \\ 2 \\ 21 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 5$ **1** • Calculate angle between line and normal vector $\theta° = \cos^{-1}\left(\frac{5}{\lvert\underset{\sim}{n}\rvert\lvert\underset{\sim}{l}\rvert}\right)$ $= \cos^{-1}\left(\frac{5}{\sqrt{1534} \times \sqrt{14}}\right)$ $= 88 \cdot 0°$ **1** • Solution angle between line and plane is $90° - 88 \cdot 0° = 2 \cdot 0°$ **1**	**5**

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark		Marks
	Unit / Outcome	Marks at levels					
		A/B	C				
B.	3c Sequences & Series	5		Find the Maclaurin expansion for $f(x) = e^{\sin x}$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ as far as the x^4 term.	• Expansion for exp(x) $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$	1	
					• Expansion for sin x $\sin x \approx x - x^3$	1	
					• Substitute sin x expansion into exp(x) expansion $e^x \approx 1 + (x - x^3) + \frac{(x - x^3)^2}{2} + \frac{(x - x^3)^3}{6} + \frac{(x - x^3)^4}{24}$	1	
					• Simplify $e^{\sin x} \approx 1 + x - x^3 + \frac{x^2 - 2x^4}{2} + \frac{x^3 + \ldots}{6} + \frac{x^4 + \ldots}{24}$	1	
					• Solution $e^{\sin x} \approx 1 + x + \frac{x^2}{2} - \frac{5x^3}{6} - \frac{23x^4}{24}$	1	5

No	Analysis			Question	Illustrations of evidence for awarding each mark		Marks
	Unit / Outcome	Marks at levels					
		A/B	C				
C.	3b Matrices		4	Given that for matrix A, $A^2 = 5A - 2I$ where I is the corresponding identity matrix, find the integers x and y such that $$A^4 = xA + yI$$	• Square expression for A squared $A^4 = (5A - 2I)(5A - 2I)$ $= 5A^2 - 10AI - 10AI + 4I^2$	1	
					• Know that $AI = A$ and that I squared = I $5A^2 - 20A + 4I$	1	
					• Substitute in expression for A squared $5(5A - 2I) - 20A + 4I$	1	
					• Solution $5A - 6I \quad \therefore x = 5 \; and \; y = -6$	1	4

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
D.	3d Differential Equations	5	2	Obtain the general solution of the differential equation $$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-15y=8\cos x$$	• Form and solve the auxiliary equation $Aux.Eq^n:\ m^2+2m-15=0\ \therefore m=-5,\ 3$ **1** • Derive the complementary function $Comp.Func^n:\ y=Ae^{-5x}+Be^{3x}$ **1** • Define the particular integral and differentiate twice $Part.Intg:\ y=C\cos x+D\sin x$ $\frac{dy}{dx}=-C\sin x+D\cos x$ $\frac{d^2y}{dx^2}=-C\cos x-D\sin x$ **1** • Substitute particular integral into original differential equation $(-C\cos x-D\sin x)$ $+2(-C\sin x+D\cos x)$ $-15(C\cos x+d\sin x)=8\cos x$ **1** • Equate coefficients $2D-16C=8\ and\ -16D-2C=0$ **1** • Find values for constants $C=-\frac{8}{17}\ and\ D=\frac{1}{17}$ **1** • State the general solution $Gen.Sol^n:$ $y=Ae^{-5x}+Be^{3x}-\frac{8}{17}\cos x+\frac{1}{17}\sin x$ **1**	**7**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
D.(Cont)			3	Hence find the particular solution given that $\frac{dy}{dx} = 0$ and $y = 2$, when $x = 0$.	• Differentiate general solution $\frac{dy}{dx} = -5Ae^{-5x} + 3Be^{3x} + \frac{8}{17}\sin x - \frac{1}{17}\cos x$ **1** • Substitute $x = 0$ into general solution and into its derivative When $x = 0,\ y = A + B - \frac{8}{17} = 2$ $and\ \frac{dy}{dx} = -5A + 3B - \frac{1}{17} = 0$ **1** • Solve to find constants and state particular solution $A = \frac{125}{136}\ and\ B = \frac{211}{136}$ so particular solution is $y = \frac{125}{136}e^{-5x} + \frac{211}{136}e^{3x} - \frac{8}{17}\cos x + \frac{1}{17}\sin x$ **1**	**3**

Total 28 marks

[END OF MARKING SCHEME FOR ADDITIONAL QUESTIONS]